

General Quantum State Swap: an XY model analysis

Ben-Qiong Liu,¹ Lian-Ao Wu,^{2,3} Bin Shao¹, and Jian Zou¹

¹Key Laboratory of Cluster Science of Ministry of Education,
and School of Physics, Beijing Institute of Technology, Beijing 100081, China

²Department of Theoretical Physics and History of Science,
The Basque Country University (EHU/UPV), PO Box 644, 48080 Bilbao, Spain

³IKERBASQUE, Basque Foundation for Science, 48011 Bilbao, Spain

We consider an exact state swap, defined as the swap between two quantum states $|A\rangle$ and $|B\rangle$ in the Hilbert space of a quantum system. We show that, given an arbitrary Hamiltonian dynamics, there is a straightforward approach to calculating the probability of the occurrence of an exact state swap, by employing an exchange operator P_{AB} . For a given dynamics, the feasibilities of proposed quantum setups, such as quantum state amplifications and transfers can be evaluated. These setups are only distinguished by different forms of P_{AB} , which easily lead to innovative designs of quantum setups or devices. We illustrate the method with the isotropic XY model, whose unnoticed features are revealed.

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Introduction.— One of challenges in quantum control (QC) and quantum information processing (QIP) is to reliably transmit known or unknown quantum states from one subspace to another in the whole Hilbert space of a quantum system. The transmission may be from a state in one information processor A to another in B or between different energy levels, and can often be characterized by a swap between two remote or local states. Examples include a swap between two neighbour or distant states, quantum-state amplification[1–4], quantum entanglement transfer[5], entanglement routers[6], or long distance quantum communications via optical fiber. These swaps may be conveniently processed in terms of local spin couplings such as the XY interaction, where no dynamical control is required. One of these swaps, quantum state transfer, has been studied extensively. These studies have developed analytic and numerical methods to evaluate the feasibility of a given dynamics.

Typically, quantum state transfer through spin chains is achieved by placing a spin state at one end of the chain and waiting for a specific amount of time to let this state evolve naturally under spin dynamics and propagate to the other end. When the quantum state propagates across the spin chain, it often loses its integrity [7, 8]. The crucial task is to calculate the fidelity of quantum transmission, which is defined by the overlap between the received and expected states in the receiver, $F = \sqrt{\langle\phi(0)|\rho(t)|\phi(0)\rangle}$. Here $|\phi(0)\rangle$ is a state at the receiver with the same form as the initial state, $\rho(t)$ is the reduced density matrix of the receiver at time t and is obtained by tracing over all but the receiver's sites. However, this quantity cannot always be readily obtained, and normally the complexity of numerical computations grows with the distance between the sender and receiver[9].

Here we introduce a general method to analyze the possibility of exact quantum swaps for an arbitrary Hamiltonian H and for arbitrary time τ . Without loss of generality, this paper will focus on swaps between two separate processors A and B with the same internal structure linked with media, exemplified by optical lattices, Josephson junction arrays[10].

Specifically, we construct a unitary operator W . The eigenstates of W encode all information of the possibilities of a specific state swap. The problem of solving the Schrödinger equation now becomes an eigen-problem of W . More significantly, it can even become an eigen-problem of these simple exchange operators under the eigenstates of the Hamiltonian. The method is directly applicable to swaps between two identical subspaces of a Hilbert space of an arbitrarily-given quantum system.

It should be pointed out that while the perfect fidelity of state transfer via a naturally-available interaction seems to be unattainable as shown in [11–18], it can be achieved by properly pre-engineering the coupling strengths [12]. This process has been termed as perfect state transfer and will exactly swap two states in processors A and B .

Formalism.— Our first step is to employ the $A \Leftrightarrow B$ permutation operator E_{AB} to swap all states in processors A and B , such that the quantum information is transferred from A to B , and vice versa. The permutation operator can be expressed explicitly by

$$E_{AB} = \sum_{\alpha\beta} (|\beta_A\rangle\langle\alpha_A|) \otimes (|\alpha_B\rangle\langle\beta_B|), \quad (1)$$

where $\alpha, \beta = 1, 2, \dots, 2^K$ for K qubits located in processors A and B and $E_{AB}^2 = 1$. $|\alpha(\beta)_{A(B)}\rangle$ refers to a state $|\alpha(\beta)\rangle$ in processor $A(B)$. We then apply a gate V_B locally on processor B to obtain a desired state $|B\rangle$. The total operator for the two actions reads

$$P_{AB} = V_B E_{AB}, \quad (2)$$

with $P_{AB} P_{AB}^\dagger = 1$. It can serve as a building block in innovative designs of quantum devices [19]. A quantum state transfer is the simplest case with $V_B = 1$. The remote amplification of a quantum state $|000\rangle_A$ to $|111\rangle_B$ can be achieved by applying the permutation operator then followed by $V_B = \sigma_x \sigma_x \sigma_x$. The remote-controlled exchange $|01\rangle_A \rightarrow |10\rangle_B$ is realized by setting $V_B = \exp(-i\pi\sigma_x\sigma_x/4)$.

We now introduce the above-mentioned operator $W(\tau) = P_{AB}U(\tau)$, which is unitary

$$W^\dagger(\tau)W(\tau) = U^\dagger(\tau)P_{AB}^\dagger P_{AB}U(\tau) = 1. \quad (3)$$

As any unitary operator, the operator $W(\tau)$ can be diagonalized and has a complete set of orthonormal eigenvectors $\{|\psi_m(0)\rangle\}_\tau$ and exponential eigenvalues $\{\exp(i\omega_m)\}_\tau$, where ω_m are real. A vector $|\psi_m(0)\rangle$ in the set obeys the eigenequation

$$W(\tau)|\psi_m(0)\rangle = \exp(i\omega_m)|\psi_m(0)\rangle. \quad (4)$$

It can be rewritten in a more interesting form

$$U(\tau)|\psi_m(0)\rangle = \exp(i\omega_m)P_{AB}|\psi_m(0)\rangle, \quad (5)$$

where the unitary condition is used. The left-hand side of Eq.(5) is the wave function $|\psi_m(\tau)\rangle$ of the system initially prepared at the eigenstate $|\psi_m(0)\rangle$. In the case that the eigenstate $|\psi_m(0)\rangle$ is a product state

$$|\psi_m(0)\rangle = |A\rangle \otimes |C\rangle, \quad (6)$$

where $|C\rangle$ denotes a state outside processor A , we can obtain

$$\begin{aligned} |\psi_m(\tau)\rangle &= \exp(i\omega_m)P_{AB}|A\rangle \otimes |C\rangle \\ &= \exp(i\omega_m)|B\rangle \otimes |C'\rangle, \end{aligned} \quad (7)$$

The use of those eigenstates $|\psi_m(0)\rangle$ as the initial states will lead to the exact quantum transmission $|A\rangle \Leftrightarrow |B\rangle$. The details of states $|C\rangle$ and $|C'\rangle$ play no roles in this process. This is an ideal situation and usually only happens for special families of Hamiltonians as in the perfect state transfers.

Of particular interest is that, for a given Hamiltonian initially prepared in any state $|\phi(0)\rangle$, our method can obtain the probability of achieving exact quantum swaps $|A\rangle \Leftrightarrow |B\rangle$. The prescription is the numerical diagonalization of the operator $W(\tau)$ such that we can calculate the overlap between the initial state $|\phi(0)\rangle$ and the eigenstates $|\psi_m(0)\rangle$, which is the probability of exact quantum swap, $p_m = |\langle\phi(0)|\psi_m(0)\rangle|^2$. If the overlap $|\langle\phi(0)|\psi_m(0)\rangle|^2 \rightarrow 1$, an exact quantum transmission or swap occurs. The swap is partial when $1 > |\langle\phi(0)|\psi_m(0)\rangle|^2 > 1/2$. This overlap is a direct indicator of quantum quantum swaps between A and B .

A case analysis: The isotropic XY model.— Spin chains are of great interest in quantum information science since they are natural candidates for quantum channels in atomic scales. The sender can transfer a quantum state to the receiver via a naturally available Hamiltonian and does not require manipulation or control over the chains. The Hamiltonian of the isotropic Heisenberg XY chain reads

$$H = -J \sum_{j=1}^N (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y), \quad (8)$$

where the uniform interaction strength J between nearest neighbour sites is taken as $J = 1$ for simplicity, and

S_j^γ ($\gamma = x, y, z$) are the spin-half operators at the j th lattice site. N is the total number of spins and is assumed to be odd for convenience. The periodic boundary conditions ($S_{N+1}^\gamma = S_1^\gamma$) are used. In addition, we consider the whole system in the "one-magnon" state, in which the number of spin-ups in the chain is one. Besides, the conclusions for the XY model in the subspaces of the zero and one magnons are applicable for the XXZ spin chain $H = J \sum_{j=1}^N (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z + h S_j^z)$ in the subspace of zero and one magnons.

The Hamiltonian (8) can be diagonalized via a Jordan-Wigner map followed by a Fourier transformation. The eigenvalues and eigenstates are [20]

$$E_m = -\cos\left(\frac{2\pi m}{N}\right), |\Phi_m\rangle = \frac{1}{\sqrt{N}} \sum_j e^{i2\pi m j/N} |\hat{j}\rangle, \quad (9)$$

where $m = 1, \dots, N$, the state $|\hat{j}\rangle = |00 \dots j \dots 0\rangle$ represents that the state of the j th site has been flipped to a spin-up state while the other spins remain spin-down and spans the one-magnon subspace.

Here we will study several cases, where we calculate the probability of the exact quantum transmission or swap. In the first case, the exchange operator P_{AB} swaps the states of corresponding spin pairs in the chain, corresponding to quantum state transfer.

A. Quantum state transfer.— In view of the great potentialities of solid-state quantum information, focus has been on implementation of quantum state transfer by spin chains. We now start with the initial state $|\phi(0)\rangle = |\hat{1}\rangle$ and let this state propagate to the r th spin after time τ , where $r = (N + 1)/2$ is the site of our receiver. The exchange operator $P_1 = |\hat{1}\rangle\langle\hat{r}| + |\hat{r}\rangle\langle\hat{1}| + P_0$ as depicted in Fig.1(a), where $P_0 = \sum_{j \neq 1, r} |\hat{j}\rangle\langle\hat{j}|$ ($j \neq 1, r$). Using the eigenstates (9), we can numerically obtain N eigenstates and eigenvalues of $W(\tau) = P_1 U(\tau)$, which are functions of τ , to seek possible values of τ such that $p_m = |\langle\psi_m(0)|\hat{1}\rangle|^2 \rightarrow 1$. Ideally, if we find that $|\langle\psi_m(0)|\hat{1}\rangle|^2 = 1$ and $|\langle\psi_m(0)|\hat{j}\rangle|^2 = 0$ (for $j \neq 1$) at a time τ , we have a perfect transfer for the state $|\hat{1}\rangle$ from the first site to the r th site at time τ . However, our numerical calculations run over all eigenstates and long time period and show that there is no exact state transmission for the XY model, as expected.

τ	0.07	0.13	0.17	0.19	0.38	0.44	0.52
ψ_m	ψ_1	ψ_1	ψ_1	ψ_2	ψ_3	ψ_5	ψ_7
p_m	0.4997	0.4989	0.4982	0.4977	0.4910	0.4879	0.4837

TABLE I: The maximal p_m and corresponding values of τ . The corresponding eigenvalues are $e^{i\omega_m} \approx -1$.

On the other hand, the conventional quantum state transfer refers to the process of transferring an unknown state, which requires at least two eigenstates of W . Ideally, if $\{|\psi_0(0)\rangle\}_\tau$ and $\{|\psi_1(0)\rangle\}_\tau$ can both perform exact state transmissions,

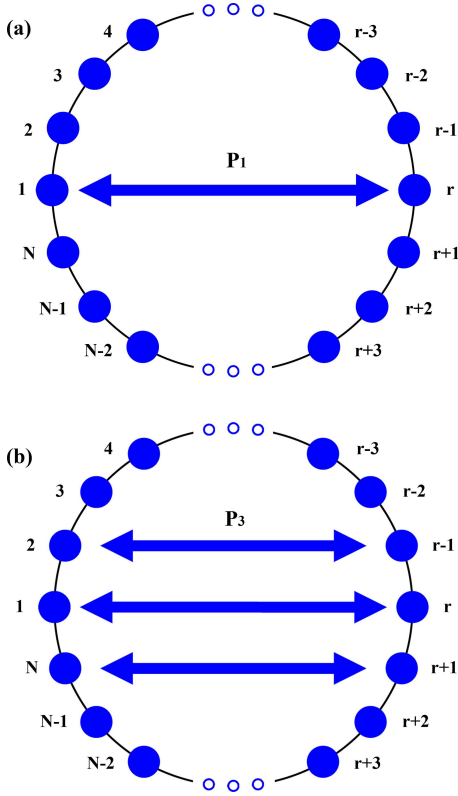


FIG. 1: Schematic of our quantum transmission protocol: (a) P_1 and (b) P_3 . The circles represent qubits.

an unknown state $a\{|\psi_0(0)\rangle\}_\tau + b\{|\psi_1(0)\rangle\}_\tau$ can be transferred perfectly if $\Delta\omega = \omega_0(\tau) - \omega_1(\tau) = 2\pi K$, with K being arbitrary integers. In the XY model, for instance, the unknown state can be $|\phi(0)\rangle = a|\widehat{0}\rangle + b|\widehat{1}\rangle$, where $|\widehat{0}\rangle$ is the zero-magnon state and a trivial eigenstate of $W(\tau)$. This is equivalent to the state $a|0\rangle + b|1\rangle$ encoded at the first site initially. We now define a joint probability of unknown state transfers, $F = \langle\phi(0)|a|\widehat{0}\rangle + b e^{i\omega_m}|\psi_m\rangle\rangle$, explicitly $F = a^2 + b^2 e^{i\omega_m} \langle\widehat{1}|\psi_m\rangle$. This joint probability may be surprisingly high, even if the probability p_m is about 0.5, for instance, $F \approx 0.97$ when $a = \sqrt{\frac{9}{10}}$, $b = \sqrt{\frac{1}{10}}$, and $F \approx 0.85$ even if $a = b = \frac{\sqrt{2}}{2}$. When $a > b$, the dominate contribution in this joint probability F is the zero-magnon state, which can always propagate from the sender to the receiver with the probability 1. Table I lists the maximum probability p_m ($m = 1, 2, \dots, N$) and corresponding time interval τ for the exchange operator P_1 .

We can also consider multi-qubit sender and receiver, for example 3 qubits as depicted in Fig.1(b), where the exchange operator is $P_3 = |\widehat{1}\rangle\langle\widehat{r}| + |\widehat{2}\rangle\langle\widehat{r+1}| + |\widehat{N}\rangle\langle\widehat{r+2}| + H.c. + P'_0$, with $P'_0 = \sum_j |\widehat{j}\rangle\langle\widehat{j}|$, ($j \neq 1, 2, r-1, r, r+1, N$). This operator swaps three pairs of spins while keeping other sites intact. An exact quantum swap means $p_3 = \sum_{j=1,2,N} |\langle\psi_m(0)|\widehat{j}\rangle|^2 = 1$, where the state of processor A composed of three spins (1st, 2nd, and N th) would ideally

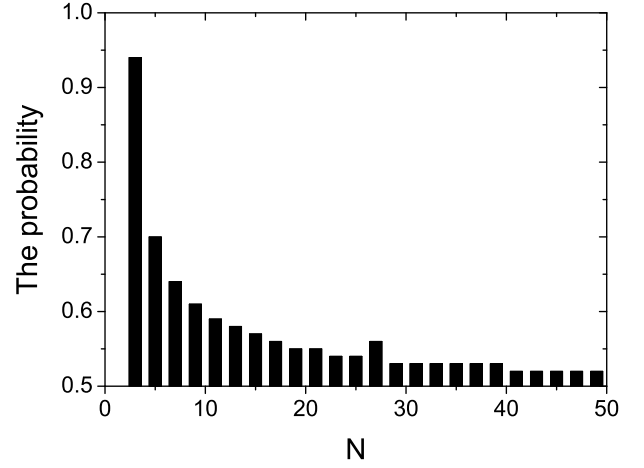


FIG. 2: The probability of an exact quantum transmission for P_{All} as a function of chain size N .

propagate to the targeted processor B . Our numerical calculation shows that the results of p_3 remain almost the same as those in one-qubit processor.

It is a conjecture that the probability of an exact quantum transmission might be improved when the number of spin pairs in processors is increased. We construct the exchange operator that swaps all the spin sites in pairs $P_{All} = (\sum_1^{M_1} |\widehat{j}\rangle\langle\widehat{r+1-j}| + \sum_{r+1}^{M_2} |\widehat{j}\rangle\langle\widehat{N+r+1-j}|) + H.c.$, where $M_1 = (N+1)/4$ and $M_2 = (3N-1)/4$ when $(N-3)/2$ is even, while $M_1 = (N-1)/4$, $M_2 = (3N+1)/4$ when $(N-3)/2$ is odd. We calculate the probability $\sum_1^N |\langle\psi_m(0)|\widehat{j}\rangle|^2$ as a function of chain size N in Fig.2. The probability decays with the numbers of spins. Our numerical calculations show that the probabilities seem to increase few percent for a given site number in one-qubit, three-qubit or multi-qubit processors but not much.

Although the isotropic spin-half XY model seems not to be good candidate for accessing exact quantum transmissions or swaps, it is interesting to note that it may be good enough for perfect state transfer when $a > b$ holds. In other words, the XY model seems to a good candidate of high-quality quantum state transfer for a family of *unknown* states, in particular states with $a > b$.

B. Remote entanglement.— The exchange operator can also help to study remote entanglement through the spin chain, where the receiver's spins are entangled while the sender's were not.

Consider the exchange operator $P_E = \frac{1}{\sqrt{2}}|\widehat{1}\rangle\langle\widehat{r}| + \langle\widehat{r+1}| + H.c. + P'_0$, which makes the two spins r th and $(r+1)$ th in the maximally entangled state $\frac{\sqrt{2}}{2}(|10\rangle + |01\rangle)$, with $P'_0 = \sum_j |\widehat{j}\rangle\langle\widehat{j}|$, ($j \neq 1, r, r+1$). Our numerical calculation shows that the maximum probability of exact remote entanglement is always ~ 0.5 , and almost does not decay with the increase of N . We present the probability $|\langle\phi(0)|\psi_1(0)\rangle_\tau|^2$ in Fig.3(a). It seems that the probability can be still high for a long chain (e.g., when $N = 73$,

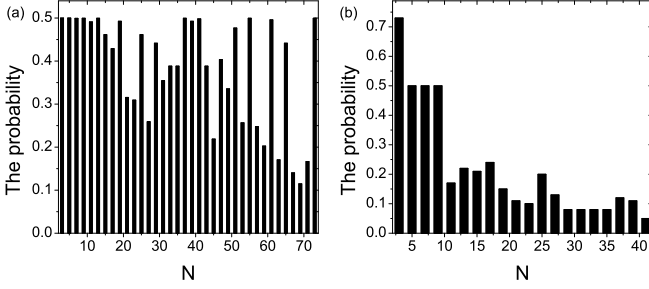


FIG. 3: The probability of an exact quantum transmission for (a) P_E , (b) P'_E as a function of chain size N .

$p \approx 0.4995$), in fact, there always exist at least one eigenstate $|\psi_m(0)\rangle$ ($m \in [1, N]$) which can have an overlap ~ 0.5 with $|\phi(0)\rangle$, independent of the chain length. This result is interesting and shows that the entangling ability of the XY model seems not to decay with the distance, which should imply an unnoticed quantumness in this model. In addition, this character of the XY model revealed here may effectively support the point of view that high fidelity state transfer over a long chain is possible[8].

Remote entanglement can also take place between remote pairs of qubits. In Fig.3(b), we take the exchange operator $P'_E = \frac{1}{\sqrt{2}}|\hat{2}\rangle(\langle\hat{1}| + \langle\hat{r}|) + H.c. + P_0^*$ which has the two spins 1st and r th maximally entangled, with $P_0^* = \sum_j |\hat{j}\rangle\langle\hat{j}|$ ($j \neq 1, 2, r$), and calculate the overlap $|\langle\phi(0)|\psi_3(0)\rangle_\tau|^2$. For the case $N = 3$, the overlap is relatively high, $p_{max} = 0.73$, so that it will give surprisingly high joint probability, for instance, $F \approx 0.96$ when $a = \frac{\sqrt{3}}{2}$. Note that except for the eigenstate marked by the subscript $m = 3$, there are another $N - 1$ choices for us in order to find the maximal overlap with the initial state $|\phi(0)\rangle$. We have numerically computed the overlaps between each eigenstate of $W(\tau)$ and the initial state, and find again that the probability 0.5 can always be achieved, no matter how large the chain is.

Similarly, we can generalize the use of the exchange operator to realize the *entanglement spreading* by the definition $P_{ES} = \frac{1}{\sqrt{N}}|\hat{1}\rangle(\langle\hat{2}| + \dots + \langle\hat{N}|) + H.c.$

Conclusions.—We have introduced a general numerical method for exact quantum transmissions, demonstrating that given an arbitrary Hamiltonian at an arbitrary time τ , the unitary operator $W(\tau) = P_{AB}U(\tau)$ can be easily numerically diagonalized such that we can obtain the probability of an exact quantum swap by calculating the overlap between the initial state of the whole system $|\phi(0)\rangle$ and the eigenstates of $W(\tau)$. We find that high-quality quantum state transfers

may be possible for a family of unknown states using the XY model. The probability for this model to create entanglement remotely does not decay with the size of the spin chain, an interesting quantum feature. In principle, it should be a fundamental feature of closed-system quantum dynamics. Experience gained from the XY model sheds light on the numerical method and will allow one to assess future directions for other applications.

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